Engineering Notes

A New Method for Numerical Integration of Airplane Equations of Motion

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Nomenclature

U = projection of airplane velocity on body X axis
α = angle of attack
θ = pitch attitude angle
β = angle of sideslip
ϕ = bank angle
ψ = heading angle
s = Laplace variable
$(\dot{\ })$ = time derivatives

It has been found in Ref. 1 that application of linear transformation methods to the integration of the airplane equations of motion can result in significant savings of computer time over more conventional methods. One of these methods, named the approximate inverse Laplace method, is applied below to the integration of rigid airplane equations of motion. These equations form a set of six simultaneous nonlinear differential equations.

By expressing these equations consistently in terms of the dependent variables U, α, θ, β, ϕ, and ψ, it is possible to write these equations in the following form:

Longitudinal (Body Axes)

$$X:\quad a_1U + a_2\dot{U} + a_3\alpha + a_4\dot{\alpha} + a_5\dot{\theta} = a_7 \tag{1a}$$

$$Z:\quad b_1U + b_3\alpha + b_4\dot{\alpha} + b_5\dot{\theta} = b_7 \tag{1b}$$

$$M:\quad c_3\alpha + c_4\dot{\alpha} + c_5\dot{\theta} + c_6\ddot{\theta} = c_7 \tag{1c}$$

Lateral-Directional (Body Axes)

$$Y:\quad d_1\beta + d_2\dot{\beta} + d_3\phi + d_5\dot{\psi} = d_7 \tag{2a}$$

$$L:\quad e_1\beta + e_3\dot{\phi} + e_4\ddot{\phi} + e_5\dot{\psi} + e_6\ddot{\psi} = e_7 \tag{2b}$$

$$N:\quad f_1\beta + f_3\dot{\phi} + f_4\ddot{\phi} + f_5\dot{\psi} + f_6\ddot{\psi} = f_7 \tag{2c}$$

It must be understood that in Eqs. (1) and (2) all coefficients are not only time variable, but also functions of the dependent variables. These functions, although complicated algebraically, are found automatically when writing the equations in the foregoing form. They are not presented in this article, because of the various forms they can take depending on the types of approximations (if any) used in the definitions of quantities involving Euler angles and relations between α, β and velocity components.

The fact is stressed, however, that even in the most rigorous development of the equations of motion, the general form expressed by Eqs. (1) and (2) is still applicable.

The numerical integration method outlined below is based on the assumption that a small interval of time Δt can be selected such that the coefficients a_i through f_i in Eqs. (1) and (2) may be considered as constants during this time interval.

With this assumption it is then possible to take the one-sided Laplace transformation of Eqs. (1) and (2). This transformation is taken for nonzero initial conditions. The transforms of a_7 through f_7 are written as a_7/s through f_7/s. These quantities will in general contain forcing functions. As an example, the transform of Eq. (1c) is

$$(c_3 + sc_4)\alpha(s) + (c_5s + c_6s^2)\theta(s) = c_7/s + c_4\alpha(0^+) + (c_5 + sc_6)\theta(0^+) + c_6\dot{\theta}(0^+) \tag{3}$$

Applying the familiar methods of linear algebra, it is evident that solutions of the transformed dependent variables can be characteristically written as

$$X(s) = N(s)/D(s) \tag{4}$$

where the order of the $D(s)$ polynomial is higher than the order of the $N(s)$ polynomial. For the case of Eqs. (1), $X(s)$ stands for $U(s)$, $\alpha(s)$, or $\theta(s)$, whereas for the case of Eqs. (2), $X(s)$ stands for $\beta(s)$, $\phi(s)$, or $\psi(s)$.

It is possible to expand (4) as follows:

$$X(s) = \frac{N(s)}{D(s)} = \frac{A_ns^n + A_{n-1}s^{n-1} + \ldots + A_0}{B_ms^m + B_{m-1}s^{m-1} + \ldots + B_0} \qquad m > n,\ B_m \neq 0 \tag{5}$$

$$= C_1s^{-1} + C_2s^{-2} + C_3s^{-3} + \ldots + \text{remainder}$$

The coefficients A_i and B_i are easily defined in terms of a_i through f_i. This is achieved by expansion of the numerator and denominator determinants for each transformed variable. The coefficients C_i are found from A_i and B_i by means of the recurrence formula

$$C_i = \frac{1}{B_m}\left[A_{n-i+1} - \sum_{j=0}^{m=1} B_jC_{i-m+j}\right] \tag{6}$$

Having defined the relations between the coefficients C_i and the coefficients a_i through f_i, the inverse of $X(s)$ yields the solution of $X(t)$ valid during the interval Δt:

$$X(t) = C_1 + C_2t + C_3(t^2/2!) + C_4(t^3/4!) + \ldots \tag{7}$$

In the case of airplane equations of motion, a time interval Δt of approximately 0.1 sec is usually selected. In that case it is found that (7) converges rapidly. In fact, no more than seven terms have to be carried to obtain good accuracy. At the end of the time interval the coefficients a_i through f_i must be recalculated in terms of the dependent variables. Since many of these coefficients contain time derivatives of the dependent variables U, α, θ, β, ϕ, and ψ, expressions for these must be developed also. It is seen that such may be done by differentiating (7) to yield

$$\dot{X}(t) = C_2 + C_3t + C_4(t^2/2!) + C_5(t^3/3!) + \ldots \tag{8}$$

$$\ddot{X}(t) = C_3 + C_4t + C_5(t^2/2!) + \ldots \tag{9}$$

In applications where very long time histories must be computed, it is possible to perform one or more iterations. In these iterations, the dependent variables and their derivatives occurring in the coefficients a_i through f_i can be replaced by corresponding values halfway between t and $t + \Delta t$. These "average" values are found by summing initial conditions and end conditions as obtained from (7) and dividing by two.

It is clear from the foregoing that during any specific time interval Δt the equations of motion are treated as an uncoupled linear set of longitudinal and lateral-directional

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equations. The coupling is taken care of at the end of each time interval by a recalculation of the coefficients a_i through f_i. Variations of aerodynamic coefficients with angle of attack, angle of sideslip, and Mach number can be accounted for by the usual table-look-up procedures.

The foregoing described numerical integration procedure has turned out to be considerably faster than conventional integration procedures. The method shows promise in the application of digital computers to real time simulation problems. When applied to studies of airplane dynamic response to externally or internally generated disturbances, significant savings in computer time are obtained. In addition, the programing involved in this method of integration is very straightforward.

Reference

[1] Roskam, J. and Ostrand, R. A., "An application of transform techniques to the solution of rigid airplane equations of motion," Boeing Document D6-2242 (August 1964).

Technical Comments

Reply by Author to Second Comment by I. L. Ashkenas

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Replying to Mr. Ashkenas' second comment on the pilot induced oscillation (PIO) results of Ref. 1 is somewhat of a challenge. The major portion of his contentions and the philosophy behind his arguments, which are contained in detail in Ref. 2, have to date received very limited circulation and certainly cannot be considered common nor a priori knowledge to the readers of this journal. My reply will, therefore, be constrained to those few comments that have been made in full view of the reader, and will ignore the implications derived from Ref. 2. However, when Ashkenas sees fit to put forth his dissertation for general consumption, I will be most happy to make public my sentiments regarding the relevancy of its contents in analyzing the PIO.

Ashkenas first contends[3] that the results from the NASA sponsored investigation,[4] which was conducted on the same dynamic simulator as the investigation under discussion, did not show PIO tendencies for similar tasks and similar vehicle frequency, damping, and control system characteristics, the implication being that the differences between the $1/T_{\theta_2}$ values used in the two studies resulted in PIO tendencies being present only in the subject study.

The facts are that neither investigation indicated any PIO tendencies during the terrain-following task for any combination of aircraft characteristics. It was only during the particular portion of the subject investigation[1] specifically designed for the PIO evaluation that PIO tendencies were apparent. The NASA study conducted no such PIO evaluation.

The second point, which is really scraping the bottom of the barrel, is that my instructions to the pilots to maneuver the aircraft as they would while making corrections in close formation flying or in any tight spot where precision control of load factor or *pitch attitude* (Ashkenas' *italics*) is critical, suggests that I feel pitch attitude control is important. Since I have already stated this to be the case[5] (but not for the situation under discussion) I see no reason to belabor the point. I was simply endeavoring to convey to the pilots that they were to fly a "tight loop" on the longitudinal control and I feel that they got the message. Thus, the inclusion of pitch attitude in the pilot's instructions was no more significant than the omission of altitude, rate of climb, pitch rate, and pitch acceleration as possible loop closures.

Next, Ashkenas deftly points out that the pilots certainly could have been tracking pitch attitude during the PIO evaluation since Ref. 4 states that the all-altitude indicator (AAI) was used for pitch corrections, that the motion of the "eight ball" was as expected, and that the short period characteristics must have been distinguishable because the airframe was rated correctly. These comments are no different than the comments received during the study in question, and I agree that it is possible. However, I feel it is highly unlikely, since both the load factor and rate of climb indicators would be more sensitive by an order of magnitude than the AAI. Also, just to set the record straight, the conclusion that the longitudinal dynamics of the basic aircraft used in Ref. 4 was "marginally satisfactory" was not obtained from that investigation but rather from the results of the investigation in question here.

Now for a final comment pertaining to Ashkenas' report[2] that he kindly forwarded for my private consumption: I found the information contained therein to be highly significant, particularly the discussion on the "cause and cure" of PIO in the Northrop T-38A airplane. The "stick free" response data quoted for this "real airplane" at a PIO sensitive condition was $1/T_{\theta_2} = 3.18$ and $2\zeta\omega_n = 1.96$, resulting in $2\zeta\omega_n - 1/T_{\theta_2}$ being negative. (The reader is reminded that $2\zeta\omega_n - 1/T_{\theta_2}$ being negative is a major bone of contention.) Lo and behold this is the very same combination of characteristics (i.e., $1/T_{\theta_2} = 3.22$, $2\zeta\omega_n = 1.26$, and $2\zeta\omega_n - 1/T_{\theta_2}$ being negative) used in the simulation investigation under discussion and considered by Ashkenas[6] to be the result of "a common deficiency in ground and flight simulation that can easily be avoided when appreciated" and "but one example of improper simulation." Since the range of variables covered in the simulation program does encompass the characteristics of the T-38 as well as most other aircraft, it would appear that the subject simulation study is not quite as deficient and improper as Ashkenas would lead us to believe. The fact that the results in question correctly diagnose the PIO tendencies of the T-38A and the A5A[7] should lend some degree of confidence to their application.

In conclusion, the foregoing rebuttal is the best that can be done at this time, and under most circumstances I would consider the matter closed. However, in answer to criticisms

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